The application of selection principles in the study of the properties of function spaces ☆

Alexander V. Osipov

Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ural State University of Economics, Yekaterinburg, Russia

Abstract

For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence.

In this paper we prove that:

• if every finite power of X is Lindelëof then $C_p(X)$ to be strongly sequentially separable iff X is γ-set.

• $B_\alpha(X)$ — functions Baire class α ($1 < \alpha \leq \omega_1$) on a Tychonoff space X with the pointwise topology — to be sequentially separable iff there exists a Baire isomorphism class α from a space X onto a σ-set.

• $B_\alpha(X)$ is strongly sequentially separable iff $iw(X) = \aleph_0$ and X is a Z^α-cover γ-set for $0 < \alpha \leq \omega_1$.

• there is a consistent example of a set of reals X, such that $C_p(X)$ is strongly sequentially separable, but $B_1(X)$ is not strongly sequentially separable.

• $B(X)$ is sequentially separable, but is not strongly sequentially separable for a $\mathfrak{b}$-Sierpiński set X.

Keywords: strongly sequentially separable, sequentially separable, function spaces, selection principles, Gerlits-Nage γ property, Baire function, σ-set, $S_1(\Omega, \Gamma)$, $S_1(B_\Omega, B_\Gamma)$, γ-set, C_p space, $\mathfrak{b}$-Sierpiński set
2000 MSC: 37F20, 26A03, 03E75, 54C35

Email address: OAB@list.ru (Alexander V. Osipov)

1. Introduction

Many topological properties are defined or characterized in terms of the following classical selection principles given in a general form in [33]. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \omega)$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \omega)$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \omega\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \omega)$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \omega)$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\omega} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{B}$.

The papers [16, 17, 33, 34, 38] have initiated the simultaneous consideration of these properties in the case where $\mathcal{A}$ and $\mathcal{B}$ are important families of open covers of a topological space X.

An open cover $\mathcal{U}$ of a space X is:

- an *ω-cover* if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$.
- a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

For a topological space X we denote:

- $\mathcal{O}$ — the family of open covers of X;
- Ω — the family of open ω-covers of X;
- Ω^{ω}_{cz} — the family of countable cozero ω-covers of X;
- Γ — the family of open γ-covers of X;
- B_Ω — the family of countable Baire (Borel for a metrizable X) ω-covers of X;
- B_Γ — the family of countable Baire (Borel for a metrizable X) γ-covers of X.

In this paper we prove that if every finite power of X is Lindelëof then $C_p(X)$ to be strongly sequentially separable iff X is γ-set (X satisfies $S_1(\Omega,\Gamma)$).

In Section 4 we prove that the space $B_\alpha(X)$ of functions Baire class α $(1 < \alpha \leq \omega_1)$ on a Tychonoff space X with the pointwise topology to be sequentially separable iff there exists a Baire isomorphism class α from a space

X onto a σ-set. Also we get criterion of strongly sequentially separableness of a space $B_\alpha(X)$ $(0 < \alpha \leq \omega_1)$.

2. Main definitions and notation

We will be denoted by

• $B_\alpha(X)$ a set of all functions of Baire class α for $\alpha \in [0, \omega_1]$ defined on a Tychonoff space X, provided with the pointwise convergence topology.

In particular:

• $C_p(X) = B_0(X)$ a set of all real-valued continuous functions $C(X)$ defined on a Tychonoff space X.

• $B_1(X)$ a set of all first Baire class functions $B_1(X)$ i.e., pointwise limits of continuous functions, defined on a Tychonoff space X.

• $B(X) = B_{\omega_1}(X)$ a set of all Baire functions, defined on a Tychonoff space X. If X is metrizable space, then $B(X)$ be called a space of Borel functions.

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a space of the weight not greater than τ. It well-known that $iw(X) = d(B_\alpha(X))$ for $0 \leq \alpha \leq \omega_1$ (Theorem 1 in [30]). In particular, for every Tychonoff space X, $iw(X) = d(C_p(X))$ ([25]).

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If D is a countable sequentially dense subset of X then X call sequentially separable space.

Call X strongly sequentially separable if X is separable and every countable dense subset of X is sequentially dense.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set. If a set $Z = \cup_i Z_i$ where Z_i is a zero-set of X for any $i \in \omega$ then Z is called a Z_σ-set of X. Note that if a space X is a perfect normal space, then class of Z_σ-sets of X coincides with class of F_σ-sets of X.

It is well known [18], that $f \in B_1(X)$ iff $f^{-1}(G)$ — Z_σ-set for any open set G of real line $\mathbb{R}$.

Recall that the cardinal $\mathfrak{p}$ is the smallest cardinal so that there is a collection of $\mathfrak{p}$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

For $f, g \in \omega^\omega$, let $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. $\mathfrak{b}$ is the minimal cardinality of a $\leq^*$-unbounded subset of ω^ω. A set $B \subset [\omega]^\infty$ is unbounded if the set of all increasing enumerations of elements of B is unbounded in ω^ω, with respect to $\leq^*$. It follows that $|B| \geq \mathfrak{b}$. A subset S of the real line is called a Q-set if each one of its subsets is a G_δ. The cardinal $\mathfrak{q}$ is the smallest cardinal so that for any $\kappa < \mathfrak{q}$ there is a Q-set of size κ. (See [9] for more on small cardinals including $\mathfrak{p}$).

Further we use the following theorems.

Theorem 2.1. *([41]). A space $C_p(X)$ is sequentially separable iff there exist a condensation (one-to-one continuous map) $f : X \mapsto Y$ from a space X on a separable metric space Y, such that $f(U)$ — F_σ-set of Y for any cozero-set U of X.*

Theorem 2.2. *([41]). A space $B_1(X)$ is sequentially separable for any separable metric space X.*

Note that proof of this theorem gives more, namely there exists a countable subset $S \subset C(X)$, such that $[S]_{seq} = B_1(X)$.

By a *set of reals* we usually mean a zero-dimensional, separable metrizable space.

3. Continuous functions

Recall that X has the property γ: for any open ω-cover α of X there is a sequence $\beta \subset \alpha$ such that β is a γ-cover of X.

Recall that X has projectively property (γ), if every continuous second countable image of X has the property γ [5].

Note that $S_1(\Omega, \Gamma)$ is equivalent to the γ-property introduced by Gerlits and Nagy in [13]. So we will call projectively $S_1(\Omega, \Gamma)$ instead of projectively (γ).

In ([5], Theorem 54), M. Bonanzinga, F. Cammaroto, M. Matveev proved

Theorem 3.1. *The following conditions are equivalent for a space X:*

1. $X \models$ *projective* $S_1(\Omega, \Gamma)$*;*
2. *every Lindelöf image of X has property (γ);*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X) \models S_1(\Omega, \Gamma)$;*

4. *for every continuous mapping $f : X \mapsto \mathbb{R}$, $f(X) \models S_1(\Omega, \Gamma)$;*
5. *for every countable ω-cover $\mathcal{U}$ of X by cozero sets, one can pick $U_n \in \mathcal{U}$ so that every $x \in X$ is contained in all but finitely many U_n;*
6. *$X \models S_1(\Omega^{\omega}_{cz}, \Gamma)$.*

In [28], authors was proved

Theorem 3.2. *For a Tychonoff space X the following statements are equivalent:*

1. *$C_p(X)$ is strongly sequentially separable;*
2. *$X \models$ projective $S_1(\Omega, \Gamma)$ and $iw(X) = \aleph_0$.*

By Proposition 55 in [11], we have that
• every projectively (γ) space is zero-dimensional;
• every space of cardinality less than $\mathfrak{p}$ is projectively (γ);
• the projectively (γ) property is preserved by continuous images.
Then we have the next

Proposition 3.3. *Let X be a Tychonoff space with $iw(X) = \aleph_0$.*
• *If $C_p(X)$ is strongly sequentially separable, then X is zero-dimensional.*
• *If cardinality of X less than $\mathfrak{p}$, then $C_p(X)$ is strongly sequentially separable.*
• *If $C_p(X)$ is strongly sequentially separable and $h : X \mapsto Y$ is continuous mapping from X onto a space Y with $iw(Y) = \aleph_0$, then $C_p(Y)$ is also strongly sequentially separable.*

Proposition 3.4. *Suppose a space $X \models$ projective $S_1(\Omega, \Gamma)$, and $F \subseteq X$ be a Z_σ-set of X. Then $F \models$ projective $S_1(\Omega, \Gamma)$.*

Proof. Let $\mathcal{V}_n = \{V^n_s : s \in \omega\} \in \Omega^{\omega}_{cz}$ of F for each $n \in \omega$, and $F = \bigcup\limits_i F_i$, where F_i is a zero-set of X and $F_i \subset F_{i+1}$ for each $i \in \omega$.

Consider $\mathcal{V}_1 = \{V^1_s : s \in \omega\}$ and $i_0 = 1$. For each $s \in \omega$ there is F_{i_s} such that $F_{i_s} \nsubseteq V^1_s$ and $i_s > i_{s-1}$. Let $\mathcal{W}_1 = \{V^1_s \cup (X \setminus F_{i_s}) : s \in \omega\}$.

Fix $n \in \omega \setminus \{1\}$ consider $\mathcal{V}_n = \{V^n_s : s \in \omega\}$ and $i_0 = n$. For each $s \in \omega$ there is F_{i_s} such that $F_{i_s} \nsubseteq V^n_s$ and $i_s > i_{s-1}$. Let $\mathcal{W}_n = \{W^n_s := V^n_s \cup (X \setminus F_{i_s}) : s \in \omega\}$. Claim that $\mathcal{W}_n \in \Omega^{\omega}_{cz}$ of X for each $n \in \omega$. Let K be a finite subset of X. Since $X \setminus F \subset W^n_s$ for $s \in \omega$ we can assume that $K \subset F$. There exists $s \in \omega$ such that $K \subset V^n_s$. Then $K \subset W^n_s$.

Since X has the property $S_1(\Omega^\omega_{cz}, \Gamma)$ there is the sequence $\mathcal{S} = \{W^n_{s(n)} = V^n_{s(n)} \cup (X \setminus F_{i_{s(n)}}) : n \in \omega\}$ such that $W^n_{s(n)} \in \mathcal{W}_n$ and $\mathcal{S}$ is a γ-cover of X.

We claim that $\{V^n_{s(n)} : n \in \omega\}$ is a γ-cover of F.

Let P be a finite subset of F. There is $i' \in \omega$ such that $P \subset F_{i'}$. Since $\mathcal{S}$ is a γ-cover of X there is n' such that $n' > i'$ and $P \subset W^n_{s(n)}$ for each $n > n'$, hence, $P \subset V^n_{s(n)}$ for each $n > n'$.

□

Corollary 3.5. Let $C_p(X)$ be strongly sequentially separable for a Tychonoff space X, and $F \subseteq X$ be a Z_σ-set of X. Then $C_p(F)$ is strongly sequentially separable.

Theorem 3.6. *Let X be a Tychonoff space, $iw(X) = \aleph_0$ and $A \subseteq X$. The space $(X \setminus A) \bigsqcup A \models$ projective $S_1(\Omega, \Gamma)$ iff a space $X \models$ projective $S_1(\Omega, \Gamma)$, and A and $X \setminus A$ are Z_σ-sets in X.*

Proof. (1) $\Rightarrow$ (2). This implication may be proved in much the same way as Theorem 5 in [10]. Since $iw(X) = \aleph_0$ there are countable cozero family γ in X such that for each $F \in [(X \setminus A) \bigsqcup A]^{<\omega}$ there exists functionally separated (in X) subsets C_F, $D_F \in \gamma$ such that $F \subset C_F \bigsqcup D_F \subset (X \setminus A) \bigsqcup A$.

By the countable γ-property there exists F_n for $n \in \omega$ such that

$(X \setminus A) \bigsqcup A \subset \bigcup\limits_{n} \bigcap\limits_{m>n} (C_{F_m} \bigsqcup D_{F_m})$.

Since C_{F_n} and D_{F_n} are functionally separated sets in X (i.e., there is $f_n \in C(X)$ such that $f_n^{-1}(0) \supseteq C_{F_n}$ and $f_n^{-1}(1) \supseteq D_{F_n}$)

$\bigcup\limits_{n} \bigcap\limits_{m>n} f_m^{-1}(0)$ and $\bigcup\limits_{n} \bigcap\limits_{m>n} f_m^{-1}(1)$ are disjoint, and they show that $X \setminus A$ and A are Z_σ-sets in X.

The mapping $id : (X \setminus A) \bigsqcup A \mapsto X$ is continuous, hence, X is a γ-set.

(2) $\Rightarrow$ (1). Consider a countable cozero ω-cover $\alpha = \{V_k\}$ of $(X \setminus A) \bigsqcup A$. Let $A = \bigcup\limits_{i} F_i = \bigcap\limits_{j} G_j$ where F_i is a zero-set of X and G_j is an cozero-set of X for each $i, j \in \omega$. Denote $\widetilde{V_k} = V_k \cup (G_k \setminus F_k)$ for each $k \in \omega$. Note that $\{\widetilde{V_k}\}$ is a countable cozero ω-cover of X. There is a sequence $\beta = \{\widetilde{V_{k_s}}\} \subset \{\widetilde{V_k}\}$ such that β is a γ-cover of X. It follows that $\{V_{k_s}\}$ is a γ-cover of $(X \setminus A) \bigsqcup A$.

□

For $X \subseteq [0, 1]$ let $X + 1 = \{x + 1 : x \in X\}$.

Corollary 3.7. (Theorem 5 in [10]) Suppose $A \subseteq X \subseteq [0,1]$. A space $(X \setminus A) \cup (A+1)$ is a γ-set iff the space X is a γ-set and A is G_δ and F_σ in X.

Note that property (γ) is preserved by finite powers [12], but for projective $S_1(\Omega,\Gamma)$ is not the case (Example 58 in [5]).

Proposition 3.8. *Suppose $X \models$ projective $S_1(\Omega,\Gamma)$. Then $X \bigsqcup X \models$ projective $S_1(\Omega,\Gamma)$.*

Proof. Let $\mathcal{U} = \{U_i : i \in \omega\}$ be a countable ω-cover of $X \bigsqcup X$ by cozero sets. Let $X \bigsqcup X = X_1 \bigsqcup X_2$ where $X_i = X$ for $i = 1, 2$. Consider $\mathcal{V}_1 = \{U_i^1 = U_i \bigcap X_1 : X_1 \setminus U_i \neq \emptyset, i \in \omega\}$ and $\mathcal{V}_2 = \{U_i^2 = U_i \bigcap X_2 : X_2 \setminus U_i \neq \emptyset, i \in \omega\}$ as families of subsets of the space X. Define $\mathcal{V} := \{U_i^1 \bigcap U_i^2 : U_i^1 \in \mathcal{V}_1$ and $U_i^2 \in \mathcal{V}_2\}$. Note that $\mathcal{V}$ is a countable ω-cover of X by cozero sets. By Theorem 3.1, there is $\{U_{i_n}^1 \bigcap U_{i_n}^2 : n \in \omega\} \subset \mathcal{V}$ such that $\{U_{i_n}^1 \bigcap U_{i_n}^2 : n \in \omega\}$ is a γ-cover of X. It follows that $\{U_{i_n} : n \in \omega\}$ is a γ-cover of $X \bigsqcup X$.

□

Proposition 3.9. *Suppose $X \models$ projective $S_1(\Omega,\Gamma)$, A and $X \setminus A$ are Z_σ-sets in X. Then $X \bigsqcup A \models$ projective $S_1(\Omega,\Gamma)$.*

Proof. By Theorem 3.6, $(X \setminus A) \bigsqcup A \models$ projective $S_1(\Omega,\Gamma)$. Let $Y = ((X_1 \setminus A_1) \bigsqcup A_1) \bigsqcup ((X_2 \setminus A_2) \bigsqcup A_2))$ where $X_i = X$, $A_i = A$ for $i = 1, 2$. By Proposition 3.8, Y is projective $S_1(\Omega,\Gamma)$.

Define the continuous mapping $f : Y \mapsto X \bigsqcup A$ defined by f

$$f = \begin{cases} id(X_1 \setminus A_1) = X \setminus X_A \\ id(A_1) = X_A \\ id(X_2 \setminus A_2) = X \setminus X_A \\ id(A_2) = A \end{cases}$$

where $X_A \subset X$ such that $X_A = A$ and id is an identity mapping. Note that the projectively (γ) property is preserved by continuous images (Proposition 55 in [11]). Since $Y \models$ projective $S_1(\Omega,\Gamma)$ and f is a continuous mapping this implies $X \bigsqcup A \models$ projective $S_1(\Omega,\Gamma)$.

□

Corollary 3.10. Suppose X is a γ-set, A is G_δ and F_σ in X. Then $X \bigsqcup A$ is a γ-set.

Proposition 3.11. *Suppose a Tychonoff space X such that $C_p(X)$ is strongly sequentially separable, $A \subset X$ and A and $X \setminus A$ are Z_σ-sets in X. Then $C_p(X \bigsqcup A)$ is strongly sequentially separable.*

Corollary 3.12. Suppose X is a perfect normal space and $C_p(X)$ is strongly sequentially separable, A is G_δ and F_σ in X. Then $C_p(X \bigsqcup A)$ is strongly sequentially separable.

Recall that $l^*(X) \leq \aleph_0$ if every finite power of X is Lindelëof (or, by Proposition in [13], if every ω-cover of X contains an at most countable ω-subcover of X).

Theorem 3.13. *For a Tychonoff space X with $l^*(X) \leq \aleph_0$ and $n \in \omega$ the following statements are equivalent:*

1. $C_p(X)$ *is strongly sequentially separable;*
2. $C_p(X^n)$ *is strongly sequentially separable;*
3. $(C_p(X))^{\aleph_0}$ *is strongly sequentially separable;*
4. $C_p(X)$ *is separable and Frechet-Urysohn;*
5. $C_p(X^n)$ *is separable and Frechet-Urysohn;*
6. $(C_p(X))^{\aleph_0}$ *is separable and Frechet-Urysohn;*
7. $iw(X) = \aleph_0$ *and* $X \models$ *projective* $S_1(\Omega, \Gamma)$*;*
8. $iw(X) = \aleph_0$ *and* $X^n \models$ *projective* $S_1(\Omega, \Gamma)$*;*
9. $iw(X) = \aleph_0$ *and* $X \models S_1(\Omega, \Gamma)$*;*
10. $iw(X^n) = \aleph_0$ *and* $X^n \models S_1(\Omega, \Gamma)$*;*
11. $iw(X) = \aleph_0$ *and* X *has* γ *property;*
12. $iw(X^n) = \aleph_0$ *and* X^n *has* γ *property;*
13. $C_p(X, \mathbb{R}^{\aleph_0})$ *is separable and Frechet-Urysohn;*
14. $C_p(X, \mathbb{R}^{\aleph_0})$ *is strongly sequentially separable.*

Proof. By Theorem 3.2, Proposition 3.8 and Theorem II.3.2 in [3], and that $(C_p(X))^{\aleph_0}$ is homeomorphic to the space $C_p(X, \mathbb{R}^{\aleph_0})$ ([2]).

□

By Todorčević Theorem (Theorem 4 in [10]) and Theorem 3.13 we have the next

Proposition 3.14. *Assuming $\diamondsuit_{\omega_1}$ there exists a γ-set X of cardinality $\omega_1 = \mathfrak{c}$ that for every subset Y of X the space $C_p(Y)$ is strongly sequentially separable.*

4. Baire functions class α

Let X be a Tychonoff space and $C(X)$ the space of continuous real-valued functions on X. Let $B_0(X) = C(X)$, and inductively define $B_\alpha(X)$ for each ordinal $\alpha \leq \omega_1$ to be the space of pointwise limits of sequences of functions in $\bigcup\limits_{\xi<\alpha} B_\xi(X)$. Let $B^*_\alpha(X)$ be the space of bounded functions in $B_\alpha(X)$.

The functions in $B(X) = B_{\omega_1}(X) = \bigcup\limits_{\alpha<\omega_1} B_\alpha(X)$ are called Baire functions or, if X is metrizable, Borel functions.

The Baire sets of X of multiplicative class α, denoted by $Z_\alpha(X)$, are defined to be the zero sets of functions in $B^*_\alpha(X)$. Those of additive class α, denoted by $CZ_\alpha(X)$, are defined as the complements of sets in $Z_\alpha(X)$. $Z_{\omega_1}(X) = \bigcup\limits_{\alpha<\omega_1} Z_\alpha(X)$. Finally, those of ambiguous class α, denoted by $A_\alpha(X)$, are the sets which are simultaneously in $Z_\alpha(X)$ and $CZ_\alpha(X)$.

With the set-theoretic operations of unions and intersection, $A_\alpha(X)$ is a Boolean algebra for each $\alpha \leq \omega_1$. By the Lebesgue-Hausdoff Theorem (Theorem 6.1.1 in [15]), for each $\alpha < \omega_1$

$Z_{\alpha+1}(X) = (CZ_\alpha(X))_\delta$, and $CZ_{\alpha+1}(X) = (Z_\alpha(X))_\sigma$; and if λ is a limit ordinal, then $Z_\lambda(X) = (\bigcup\limits_{\xi<\lambda} CZ_\xi(X))_{\sigma\delta}$ and $CZ_\lambda(X) = (\bigcup\limits_{\xi<\lambda} Z_\xi(X))_{\delta\sigma}$.

It is well known ([18]), that $f \in B_\alpha(X)$ iff $f^{-1}(G) \in CZ_\alpha(X)$ for any open set G of real line $\mathbb{R}$.

In [27], Osipov and Pytkeev have established criterion for $B_1(X)$ to be sequentially separable.

Theorem 4.1. *(Osipov, Pytkeev) A function space $B_1(X)$ is sequentially separable iff there exists a bijection $\varphi : X \mapsto Y$ from a space X onto a separable metrizable space Y, such that*

1. *$\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y;*
2. *$\varphi(T)$ — F_σ-set of Y for any zero-set T of X.*

Corollary 4.2. A function space $B_1(X)$ is sequentially separable iff there exists a Baire isomorphism class 1 from a space X onto a separable metrizable space.

Theorem 4.3. *For any $1 < \alpha \leq \omega_1$, a function space $B_\alpha(X)$ is sequentially separable iff there exists a bijection $\varphi : X \mapsto Y$ from a space X onto a separable metrizable space Y, such that*

1. *$\varphi^{-1}(U) \in CZ_\alpha(X)$ for any open set U of Y;*
2. *$\varphi(T)$ — F_σ-set of Y for $T \in (\bigcup\limits_{\xi<\alpha} Z_\xi(X))_\delta$.*

Proof. (1) $\Rightarrow$ (2). Let $B_\alpha(X)$ be sequentially separable, and S be a countable sequentially dense subset of $B_\alpha(X)$. Consider the topology τ generated by the family $\mathcal{P} = \{f^{-1}(G) : G$ is an open set of $\mathbb{R}$ and $f \in S\}$. A space $Y = (X, \tau)$ is a separable metrizable space because S is a countable dense subset of $B_\alpha(X)$. Note that a function $f \in S$, considered as mapping from Y to $\mathbb{R}$, is a continuous function. Let φ be the identity mapping from X on Y.

We claim that $\varphi^{-1}(U) \in CZ_\alpha(X)$ for any open set U of Y.

Note that $CZ_\alpha(X)$ is closed under a countable unions and a finite intersections. It follows that it is sufficient to prove for any $P \in \mathcal{P}$. But $\varphi^{-1}(P) \in CZ_\alpha(X)$ because $f \in S \subset B_\alpha(X)$.

Let $T \in (\bigcup\limits_{\xi<\alpha} Z_\xi(X))_\delta \subset A_\alpha(X)$ and h be a characteristic function of T. Since $T \in A_\alpha(X)$, $h \in B_\alpha(X)$ (see Theorem 1, §31 in [18]).

There exists $\{f_n\}_{n\in\omega} \subset S$ such that $\{f_n\}_{n\in\omega}$ converges to h. Since $S \subset C_p(Y)$, $h \in B_1(Y)$ and, hence, $h^{-1}(\frac{1}{2}, \frac{3}{2}) = T$ is a Z_σ-set of Y. (Note that if a space Z is a perfect normal space, then class of Z_σ-sets of Z coincides with class of F_σ-sets of Z).

(2) $\Rightarrow$ (1). Let φ be a bijection from X on Y satisfying the conditions of theorem. Then $h = f \circ \varphi \in B_\alpha(X)$ for any $f \in C(Y)$ ($h^{-1}(G) = \varphi^{-1}(f^{-1}(G)) \in CZ_\alpha(X)$ for any open set G of $\mathbb{R}$).

Moreover, $g = f \circ \varphi^{-1} \in B_1(Y)$ for any $f \in B_\alpha(X)$ because of $\varphi(Z)$ is a F_σ-set of Y for any $Z \in CZ_\alpha(X)$. Define a mapping $F : B_\alpha(X) \mapsto B_1(Y)$ by $F(f) = f \circ \varphi^{-1}$.

Since φ is a bijection, $C(Y)$ embeds in $F(B_\alpha(X))$ i.e., $C(Y) \subseteq F(B_\alpha(X)) \subseteq B_1(Y) \subseteq B_\alpha(Y) \subseteq F(B_\alpha(X))$. Note that $F(B_\alpha(X)) = B_1(Y) = B_\alpha(Y)$ i.e., Y is a σ-set (recall that a set of reals X is σ-set if each G_δ subset of X is an F_σ subset of X [18]).

By Theorem 1 in [41], each subspace D such that $C(Y) \subset D \subset B_1(Y)$ is sequentially separable. Thus $F(B_\alpha(X))$ is sequentially separable, and, hence, $B_\alpha(X)$ is sequentially separable. $\square$

Corollary 4.4. A function space $B(X)$ is sequentially separable iff there exists a bijection $\varphi : X \mapsto Y$ from a space X onto a separable metrizable space Y, such that

1. $\varphi^{-1}(U)$ — Baire set of X for any open set U of Y;
2. $\varphi(T)$ — F_σ-set of Y for any Baire set T of X.

A bijection $f : X \mapsto Y$ between Tychonoff spaces is called a Baire isomorphism if $f(Z_{\omega_1}(X)) = Z_{\omega_1}(Y)$; and is said to be of class (α,β) (or α if $\alpha = \beta$) if $f^{-1}(Z_0(Y)) \subset Z_\alpha(X)$ and $f(Z_0(X)) \subset Z_\beta(Y)$.

If X and Y are metrizable, then f is usually called a Borel isomorphism.

Note that if X is a σ-set, then $B_1(X) = B(X)$ is sequentially separable (Theorem 2.2).

Recall that $X \subset 2^\omega$ is Sierpiński set, if it is uncountable, but for every measure zero set M, $X \bigcap M$ is countable. Sierpiński showed that the Continuum Hypothesis (CH) implies the existence of such sets.

Proposition 4.5. *(CH) Let X be a Sierpiński set. Then $B_1(X) = B(X)$ is sequentially separable.*

Proof. By Szpilrajn (Marczewski) Theorem (see in [37]), if X is a Sierpiński set, then X is a σ-set. □

Corollary 4.6. Let X be a Tychonoff space and $1 < \alpha \leq \omega_1$. A function space $B_\alpha(X)$ is sequentially separable iff there exists a Baire isomorphism of class α from a space X onto a σ-set.

Corollary 4.7. Let X be a metrizable space. A space $B(X)$ of Borel functions is sequentially separable iff there exists a Borel isomorphism from a space X onto a σ-set.

Recall that Baire order $ord(C(X)) = \min\{\beta : B_\beta(X) = B_{\beta+1}(X)\}$.

Corollary 4.8. Suppose that $B_\alpha(X)$ is sequentially separable for a Tychonoff space X and $1 < \alpha \leq \omega_1$. Then Baire order $ord(C(X)) \leq \alpha$ and $ord(C(X)) < \omega_1$.

Proof. If exists a Baire isomorphism F of class α from a space X onto a σ-set Y, then F is a Baire isomorphism F of class β for every $\beta \geq \alpha$. By

Theorem 4.3, $F(B_\alpha(X)) = B_1(Y) = F(B_\beta(X))$ for every $\beta \geq \alpha$. It follows that $ord(C(X)) \leq \alpha$.

Note that if $B(X)$ is sequentially separable, then there exists a countable sequentially dense subset $S \subset B(X)$, and, hence, $S \subset B_{\alpha'}(X)$ for some $\alpha' < \omega_1$. It follows that $B_{\alpha'}(X)$ is also sequentially separable, $B_{\alpha'}(X) = B(X)$ and $ord(C(X)) < \omega_1$.

$\square$

The basic existence result are given in the next theorem.

Theorem 4.9. *(Lebesgue, [19]) If X is a complete metric space containing a non-empty perfect subset, then for all $\alpha < \omega_1$, $B_{\alpha+1}(X) \setminus B_\alpha(X) \neq \emptyset$.*

Theorem 4.10. *(Čoban, [8], Jayne, [14]) If X is a compact Hausdorff space containing a non-empty perfect subset, then for all $\alpha < \omega_1$,*
$B_{\alpha+1}(X) \setminus B_\alpha(X) \neq \emptyset$.

Theorem 4.11. *(P.R. Meyer, [24]) If X is a compact Hausdorff space which contains no non-empty perfect subsets, then $B_1(X) = B_2(X)$.*

Note that, if X is an uncountable Polish space, then $B_1(X)$ is sequentially separable (Theorem 2.2), but, by Lebesgue Theorem 4.9 and Corollary 4.8, $B_\alpha(X)$ is not sequentially separable for $1 < \alpha \leq \omega_1$. The same is true for any uncountable analytic ($\boldsymbol{\sum_1^1}$) space X since X has a perfect subspace (see [18]).

By Theorem 4.11, if X is a metrizable compact space which contains no non-empty perfect subsets, then $B_1(X) = B(X)$ is sequentially separable.

Proposition 4.12. *There exists an example (an uncountable Polish space containing a non-empty perfect subset) a space X, such that $B_1(X)$ is sequentially separable, but $B_\alpha(X)$ is not sequentially separable for all $1 < \alpha \leq \omega_1$.*

A natural generalization of a σ-set is the notion of Q-set. A set of reals X is a Q-set if every subset of X is a relative F_σ. Assuming Martin's axiom (MA) every set of reals of cardinality less than the continuum is a Q-set (Theorems in [20] or [32]).

Proposition 4.13. *(MA) Let X be a set of reals, $|X| < \mathfrak{c}$. Then $B_1(X) = R^X$ is sequentially separable.*

Proposition 4.14. *(MA) Let X be a Tychonoff space, $iw(X) = \aleph_0$ and $|X| < \mathfrak{c}$. Then $B(X)$ is sequentially separable.*

By Theorem 9.1 in [9], $\mathfrak{b} = \min\{|X| : X$ is a separable metrizable, but it is not a σ-set$\}$. Then we have a next result.

Proposition 4.15. *Let X be a Tychonoff space, $iw(X) = \aleph_0$ and $|X| < \mathfrak{b}$. Then $B(X)$ is sequentially separable.*

On the other hand, it is consistent that there are no uncountable σ-sets, in fact, it is consistent that every uncountable set of reals has Baire order ω_1 (Theorem 22 in [22]).

Proposition 4.16. *It is consistent that there are no uncountable σ-sets, if $B_\alpha(X)$ is sequentially separable for a Tychonoff space X and $\alpha > 1$, then X is countable.*

We denote:

- $Z^\alpha = (\bigcup\limits_{\xi<\alpha} Z_\xi(X))_\delta$;
- Z^α_Ω — the family of countable Z^α ω-covers of X;
- Z^α_Γ — the family of countable Z^α γ-covers of X.

A set X is called a Z^α-cover γ-set iff every countable ω-cover of X by Z^α sets contains a γ-cover.

Being a Z^α-cover γ-set is equivalent to saying that for any ω-sequence of countable Z^α ω-covers of X we can choose one element from each and get a γ-cover of X – this is denoted $S_1(Z^\alpha_\Omega, Z^\alpha_\Gamma)$.

A standard diagonalization trick gives the following.

Proposition 4.17. *A Z^α-cover γ-set X is equivalent to $X \models S_1(Z^\alpha_\Omega, Z^\alpha_\Gamma)$.*

Proof. The proof of this is like that of the corresponding result in [13]. □

We recall concept Z_σ-mapping:

a map $f : X \mapsto Y$ be called a Z_σ-map, if $f^{-1}(Z)$ is a Z_σ-set of X for any zero-set Z of Y.

In [27], Osipov and Pytkeev have established criterion for $B_1(X)$ to be strongly sequentially separable.

Theorem 4.18. *(Osipov, Pytkeev) A function space $B_1(X)$ is strongly sequentially separable iff X has a coarser second countable topology, and for any bijection φ from a space X onto a separable metrizable space Y, such that $\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y, the space Y has the property γ.*

Theorem 4.19. *For a Tychonoff space X and $0 < \alpha \leq \omega_1$, the following statements are equivalent:*

1. *$B_\alpha(X)$ is strongly sequentially separable;*
2. *$iw(X) = \aleph_0$ and $X \models S_1(Z^\alpha_\Omega, Z^\alpha_\Gamma)$;*
3. *$iw(X) = \aleph_0$ and X is a Z^α-cover γ-set.*

Proof. (1) $\Rightarrow$ (3). Let $B_\alpha(X)$ be strongly sequentially separable space and $\alpha = \{V_i\}$ be a countable Z^α ω-cover of X. Let $S = \{h_i : i \in \omega\}$ be a countable sequentially dense subset of $B_\alpha(X)$. Consider the countable set $D = \{f_{i,j} \in B_\alpha(X) : f_{i,j} \upharpoonright V_i = h_j$ and $f_{i,j} \upharpoonright (X \setminus V_i) = 1$ for $i, j \in \omega\}$. Since $\alpha = \{V_i\}$ is Z^α ω-cover of X and S is a dense subset of $B_\alpha(X)$, the set D is a dense subset of $B_\alpha(X)$. By (1), the set D is a countable sequentially dense subset of $B_\alpha(X)$. Then there exists a sequence $\{f_{i_k,j_k}\}_{k\in\omega}$ converge to $\mathbf{0}$ such that $f_{i_k,j_k} \in D$ for every $k \in \omega$. Claim that the sequence $\{V_{i_k} : k \in \omega\}$ is a γ-cover of X. Let K be a finite subset of X and $W = [K, (-1, 1)]$ is a base neighborhood of $\mathbf{0}$, then there is $k' \in \omega$ such that $f_{i_k,j_k} \in W$ for each $k > k'$. It follows that $K \subset V_{i_k}$ for each $k > k'$. We thus get that X is a Z^α-cover γ-set.

(2) $\Leftrightarrow$ (3). By Proposition 4.17.

(2) $\Rightarrow$ (1) Let $iw(X) = \aleph_0$, $X \models S_1(Z^\alpha_\Omega, Z^\alpha_\Gamma)$ and $D = \{d_i : i \in \omega\}$ be a countable dense subset of $B_\alpha(X)$. Claim that there exists a sequence $\{d_{i_k}\}_{k\in\omega}$ such that $d_{i_k} \in D$ for every $k \in \omega$ and $\{d_{i_k}\}_{k\in\omega}$ converge to $\mathbf{0}$.

The set $\mathcal{V}_j = \{V^i_j = d_i^{-1}(-\frac{1}{j}, \frac{1}{j}) : d_i \in D\}$ is a countable ω-cover of X by Z^α sets for every $j \in \omega$. Indeed, let K be a finite subset of X and $W = [K, (-\frac{1}{j}, \frac{1}{j})]$ be a base neighborhood of $\mathbf{0}$, then there is $d \in D$ such that $d \in W$, hence, $K \subset d^{-1}(-\frac{1}{j}, \frac{1}{j})$.

By (2), there is a sequence $\{V^{i(j)}_j\}_{j\in\omega}$ such that $V^{i(j)}_j \in \mathcal{V}_j$ and $\{V^{i(j)}_j : j \in \omega\}$ is a γ-cover of X. Claim that $\{d_{i(j)}\}_{j\in\omega}$ converge to $\mathbf{0}$. Let K be a finite subset of X, $\epsilon > 0$ and $O = [K, (-\epsilon, \epsilon)]$ be a base neighborhood of $\mathbf{0}$, then there is $j' \in \omega$ such that $\frac{1}{j'} < \epsilon$ and $K \subset V^{i(j)}_j$ for $j > j'$. It follows that $d_{i(j)} \in O$ for $j > j'$.

$\square$

Recall that a set X is called a Borel-cover γ-set iff every countable ω-cover of X by Borel sets contains a γ-cover (see in [21]).

Being a Borel-cover γ-set is equivalent to saying that for any ω-sequence of countable Borel ω-covers of X we can choose one element from each and get a γ-cover of X – this is denoted $S_1(B_\Omega, B_\Gamma)$. The equivalence was proved by Gerlitz and Nagy [13] for open covers i.e., for $S_1(\Omega, \Gamma)$, but the proof works also for Borel covers as was noted in Scheepers and Tsaban [35].

Proposition 4.20. *(Scheepers, Tsaban) A Borel-cover γ-set X is equivalent to $X \models S_1(B_\Omega, B_\Gamma)$.*

Note that T. Orenshtein and B. Tsaban proved (Lemma 2.8 in [26]) the interesting

Proposition 4.21. *(Orenshtein, Tsaban) Assume that a space X is a Borel-cover γ-set. Then for each countable $A \subset B(X)$, each $f \in \overline{A}$ is a pointwise limit of a sequence of elements of A.*

In [28], authors was proved

Corollary 4.22. For a Tychonoff space X the following statements are equivalent:

1. $B(X)$ is strongly sequentially separable;
2. $iw(X) = \aleph_0$ and $X \models S_1(B_\Omega, B_\Gamma)$;
3. $iw(X) = \aleph_0$ and X is a Borel-cover γ-set.

Recall that the space Y is an α_1 space, if for each $y \in Y$, each countable family $\{A_n\}$ of sequences, each converging to y, can be amalgamated as follows: there are cofinite subsets $B_n \subset A_n$, $n \in \omega$, such that the set $B = \bigcup_n B_n$ converges to y [1].

Let Y be a metric space. A function $f : X \mapsto Y$ is a quasi-normal limit of functions $f_n : X \mapsto Y$ if there are positive reals ϵ_n, $n \in \omega$, converging to 0 such that for each $x \in X$, $d(f_n(x), f(x)) < \epsilon_n$ for all but finitely many n. A topological space X is QN-space if whenever $\mathbf{0}$ is a pointwise limit of a sequence of continuous real-valued functions on X, we have that $\mathbf{0}$ is a quasi-normal limit of the same sequence.

By Corollary 20, Corollary 21 and Theorem 32 in [39], we have

Theorem 4.23. *For a set of reals X and $0 < \alpha \leq \omega_1$, the following are equivalent.*

1. $C_p(X)$ *is an* α_1 *space.*
2. $B_\alpha(X)$ *is an* α_i *space for each* $i = \overline{1,4}$.
3. X *is a* QN *space.*
4. $X \models S_1(F_\Gamma, F_\Gamma)$.
5. $X \models S_1(B_\Gamma, B_\Gamma)$.
6. *Each Baire class* α *image of* X *in* ω^ω *is bounded;*
7. *Each Borel image of* X *in* ω^ω *is bounded.*
8. $X \models S_1(Z^\alpha_\Gamma, Z^\alpha_\Gamma)$.

Note note that $S_1(B_\Gamma, B_\Gamma)$ implies that X is σ-set (Proposition 4 in [35]). It follows that $S_1(B_\Gamma, B_\Gamma) = S_1(Z^\alpha_\Gamma, Z^\alpha_\Gamma)$.

Corollary 4.24. For a set of reals X, the following are equivalent.

1. $X \models S_1(Z^\alpha_\Gamma, Z^\alpha_\Gamma)$.
2. Every F_σ-measurable mapping $f : X \mapsto \omega^\omega$ (i.e., $f^{-1}(U) \in F_\sigma$ for open U) the image $f(X) \subset \omega^\omega$ is bounded.

Recall the definition of the weak distributive law for a family of subsets of a nonempty set (see [7]).

Let X be a nonempty set, $\mathcal{A} \subseteq \mathcal{P}(X)$ being a family of subsets. $\mathcal{A}$ is called *weakly distributive* if for any system $A_{n,m} \in \mathcal{A}$, $n, m \in \omega$ such that

$\bigcap\limits_n \bigcup\limits_m A_{n,m} = X$

there exists a function $\varphi \in \omega^\omega$ such that $\bigcup\limits_k \bigcap\limits_{n \geq k} \bigcup\limits_{m \leq \varphi(n)} A_{n,m} = X$.

Theorem 4.25. *Let* X *be a perfectly normal topological space and* $0 < \alpha \leq \omega_1$. *Then the following statements are equivalent:*

1. $B_\alpha(X)$ *is strongly sequentially separable;*
2. $iw(X) = \aleph_0$ *and* $X \models S_1(B_\Omega, B_\Gamma)$*;*
3. $iw(X) = \aleph_0$ *and* $X \models S_1(F_\Omega, F_\Gamma)$.

Proof. Note that if $X \models S_1(F_\Omega, F_\Gamma)$ then the family of closed subsets of X is weakly distributive. Then X is a σ-set (see Theorem 5.2. in [7] and [18]). □

Corollary 4.26. (see Corollary 5.4 in [7]) Let X be a perfectly normal space. If $X \models S_1(F_\Omega, F_\Gamma)$ then every subset of X is a QN-space.

Theorem 4.27. ($\mathfrak{p} = \mathfrak{c}$) *There is a consistent example of a set of reals X, such that $C_p(X)$ is strongly sequentially separable, but $B_1(X)$ is not strongly sequentially separable.*

Proof. By corollary 1 in [31] (Recław's proof assumes Martin's axiom, but the partial order used is σ-centered so that in fact $\mathfrak{p} = \mathfrak{c}$ is enough (see Theorem 44 in [35])), there exists a γ-set which can be mapped onto $[0, 1]$ by a Borel function. It follows that $X \models S_1(\Omega, \Gamma)$, but X has not the property $S_1(B_\Omega, B_\Gamma)$. By Theorem 3.13 and Theorem 4.25, $C_p(X)$ is strongly sequentially separable, but $B_1(X)$ is not strongly sequentially separable. $\square$

Corollary 4.28. ($\mathfrak{p} = \mathfrak{c}$) There is a consistent example of a set of reals X, such that $X \models S_1(\Omega, \Gamma)$, but X has not the property $S_1(F_\Omega, F_\Gamma)$.

For an uncountable cardinal number κ a set of real numbers is a κ-Sierpiński set if it has cardinality at least κ, but its intersection with each set of Lebesgue measure zero is less than κ. Note that every $\mathfrak{b}$-Sierpiński set has property $S_1(B_\Gamma, B_\Gamma)$ and, hence, it is a σ-set. Since sets of real numbers having property $S_1(\mathcal{O}, \mathcal{O})$ have measure zero, no $\mathfrak{b}$-Sierpiński set has property $S_1(\mathcal{O}, \mathcal{O})$. Hence, $\mathfrak{b}$-Sierpiński set has not property $S_1(B_\Omega, B_\Gamma)$ ([35]).

By Corollary 4.6 and Theorem 4.25, we have the next

Proposition 4.29. *Let X be a $\mathfrak{b}$-Sierpiński set. Then $B(X)$ is sequentially separable, but is not strongly sequentially separable.*

5. Open questions

Question 1. Assume that for a Tychonoff space X, $C_p(X)$ is strongly sequentially separable and $\tau < \mathfrak{p}$. Does it follow that $C_p(X, \mathbb{R}^\tau)$ is strongly sequentially separable ?

Recall that a separable space X is said to be a CDH (countable dense homogeneous) space, if for any two countable dense subsets A and B in X, there is an autohomeomorphism h of X such that $h(A) = B$.

Note that $\mathbb{R}^\kappa$ is a CDH space iff $\kappa < \mathfrak{p}$.

Clearly, that a sequentially separable CDH space is a strongly sequentially separable.

Question 2. Assume that for a Tychonoff space X, $C_p(X)$ is strongly sequentially separable and $\tau < \mathfrak{p}$. Does it follow that $C_p(X, \mathbb{R}^\tau)$ is a CDH space ?

Question 3. Assume that for Tychonoff space X, $B(X)$ is strongly sequentially separable and $\tau < \mathfrak{p}$. Does it follow that $B(X, \mathbb{R}^{\tau})$ is a CDH space ?

Question 4. Assume that for a set of reals X, $B(X)$ is strongly sequentially separable. Does it follow that $B(X^n)$ is strongly sequentially separable for any $n > 1$?

Question 5. Assume that a space X is a σ-, γ-set. Does it follow that X^n is a σ-, γ-set for any $n > 1$?

Question 6. Assume that there is a Baire isomorphism (class α) from a σ-set X onto a Tychonoff space Y. Does it follow that Y is a σ-space ?

Question 7. Does **MA** imply the existence of a σ-, γ-set of size the continuum ?

Question 8. (MA) Assume that there is a σ-set X such that X is a γ-set and $|X| = \mathfrak{c}$. Does it follow that for any $Y \subset X$, Y is a γ-set ?

Question 9. Assume that $B_{\alpha}(X)$ is not sequentially separable. Does it follow that $B_{\beta}(X)$ is not sequentially separable for $\beta > \alpha$?

Acknowledgment. The author wishes to express his Thanks to Boaz Tsaban for his inspiration for writing this paper and for the detailed answers on the numerous author's questions.

References

[1] A.V. Arhangel'skii, *The frequency spectrum of a topological space and the classification of spaces*, Soviet Math. Dokl. 13, (1972), 1186–1189.

[2] A.V. Arhangel'skii, *Structure and classification of topological spaces and cardinal invariants*, Uspekhi Mat. Nauk, 33, issue 6(204), (1978), 29–84.

[3] A.V. Arkhangel'skii, *Topological function spaces*, Moskow. Gos. Univ., Moscow, (1989), 223 pp. (Arhangel'skii A.V., *Topological function spaces*, Kluwer Academic Publishers, Mathematics and its Applications, 78, Dordrecht, 1992 (translated from Russian)).

[4] T. Banakh, M. Machura, L. Zdomskyy, *On critical cardinalities related to Q-sets*, Mathematical Bulletin of the Shevchenko Scientific Society, 11, (2014), 21–32.

[5] M. Bonanzinga, F. Cammaroto, M. Matveev, *Projective versions of selection principles*, Topology and its Applications, 157, (2010), 874–893.

[6] J. Brendle, *Generic constructions of small sets of reals*, Topology and its Applications, 71, (1996), p. 125–147.

[7] L. Bukovský, I. Recław, and M. Repický, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topology and its Applications, 41, (1991), p. 25-41.

[8] M.M. Čoban, *Baire sets in complete topological spaces*, Ukrain. Mat. Ž., (1970), 22, p. 330–342.

[9] E.K. van Douwen, *The integers and topology*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, (1984).

[10] F. Galvin, A.W. Miller, *γ-sets and other singular sets of real numbers*, Topology and its Applications, 17, (1984), p.145–155.

[11] P. Gartside, J.T.H. Lo, A. Marsh, *Sequential density*, Topology and its Applications, 130, (2003), p.75–86.

[12] J. Gerlits, *Some Properties of $C(X)$,II*, Topology and its Applications, 15, (1983), p.255–262.

[13] J. Gerlits, Zs. Nagy, *Some Properties of $C(X)$,I*, Topology and its Applications, 14, (1982), p.151–161.

[14] J.E, Jayne, *Descriptive set theory in compact spaces*, Notices Amer. Math. Soc, 17, (1970), 268.

[15] J. Jayne, C. Rogers, *K-analytic Sets, Analytic Sets*, Academic Press, London (1980).

[16] W. Just, A.W. Miller, M. Scheepers, and P.J. Szeptycki, *The combinatorics of open covers, II*, Topology and its Applications, 73, (1996), p.241–266.

[17] L. Kočinac, *Selected results on selection principles*, in: Sh. Rezapour (Ed.), Proceedings of the 3rd Seminar on Geometry and Topology, Tabriz, Iran, Jule 15-17, (2004), p. 71–104.

[18] K. Kuratowski, *Topology*, Academic Press, Vol.I, (1966).

[19] H. Lebesgue, *Sur les fonctions représentables analytiquement*, J. Math. Pures Appl., 1, (1905), p. 139–216.

[20] D.A. Martin, R.M. Solovay, *Internal Cohen extensions*, Annals of Mathematical Logic, (2), 2, (1970), p. 143–178.

[21] A.W. Miller, *A nonhereditary Borel-cover γ-set*, Real Analysis Exchange, v. 29, 2, (2003), p. 601–606.

[22] A.W. Miller, *On generating the category algebra and the Baire order problem*, Bull. Acad. Polon., 27, (1979), p. 751–755.

[23] A.W. Miller, *On the length of borel hierarchies*, Annals of Mathematical Logic, 16, (1979), p. 233–267.

[24] P.A. Meyer, *The Baire order problem for compact spaces*, Duke Math. J., 33, (1966), 33–40.

[25] N. Noble, *The density character of functions spaces*, Proc. Amer. Math. Soc. (1974), V.42, is.I.-P., 228–233.

[26] T. Orenshtein, B. Tsaban *Pointwise convergence of partial functions: The Gerlits-Nagy Problem*, Advances in Mathematics, 232, issue 1, (2013), p. 311–326.

[27] A.V. Osipov, E.G. Pytkeev, *On sequential separability of functional spaces*, Topology and its Applications, 221, (2017), p. 270–274.

[28] A.V. Osipov, P. Szewczak, and B. Tsaban, *Strongly sequentially separable function spaces, via selection principles*, to appear.

[29] E. Pearl (ed.), *Open problems in topology, II*, Amsterdam, 2007.

[30] A.V. Pestrjkov, *O prostranstvah Berovskih funktsii* (On spaces of Baire functions), Issledovanij po teorii vipuklih mnogestv i grafov, Sbornik nauchnih trudov, Sverdlovsk, Ural'skii Nauchnii Center, (1987), p. 53–59.

[31] I.Recław, *On small sets in the sense of measure and category*, Fund. Math., 133, (1989), p. 255-260.

[32] M.E. Rudin, *Martin's axiom*, in: J. Barwise, ed., Handbook of Mathematical Logic (North-Holland, Amsterdam), p. 491–501.

[33] M. Scheepers, *Combinatorics of open covers (I): Ramsey Theory*, Topology and its Applications, 69, (1996), p. 31–62.

[34] M. Scheepers, *Selection principles and covering properties in topology*, Not. Mat., 22, (2003), p. 3–41.

[35] M. Scheepers, B. Tsaban, *The combinatorics of Borel covers*, Topology and its Applications, 121, (2002), p. 357–382.

[36] J. Steprans, H.X. Zhou, *Some results on CDH – I*, Topology and its Applications, 28, (1988), p. 147–154.

[37] E. Szpilrajn (Marczewski), *Sur un problème de M. Banach*, Fund. Math., 15, (1930), p. 212–214.

[38] B. Tsaban, *Some new directions in infinite-combinatorial topology*, in: J.Bagaria, S. Todorčevic (Eds.), Set Theory, in: Trends Math., Birkhäuser, (2006), p. 225–255.

[39] B. Tsaban, L. Zdomskyy, *Hereditarily Hurewicz spaces and Arhangel'skiĭ sheaf amalgamations*, Journal of the European Mathematical Society, 12, (2012), p. 353–372.

[40] B. Tsaban, L. Zdomskyy, *Scales, fields, and a problem of Hurewicz*, Journal of the European Mathematical Society, 10, (2008), p. 837–866.

[41] N.V. Velichko, *On sequential separability*, Mathematical Notes, Vol.78, Issue 5, 2005, p. 610–614.